TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 7 2016

Section 1 Calculator Free

Rectilinear Motion (including SHM) and Statistical Inference.

STUDENT'S NAME: SOLUTIONS

DATE: Thursday 8th September **TIME:** 20 minutes **MARKS**: 23

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet.

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (6 marks)

A particle oscillates 1.2 m either side of a central position with simple harmonic motion. The period of the motion is 8 seconds.

(a) What is the maximum acceleration? [4]

S.H.M. $\Rightarrow a = \frac{d^2x}{dt^2} = -k^2x$ ✓

Given period = 8

$\Rightarrow 8 = \frac{2\pi}{k}$

$\therefore k = \frac{\pi}{4}$ ✓

$\therefore$ Max acceleration

$= k^2A$ where $A = 1.2 = \frac{6}{5}$

$= \left(\frac{\pi}{4}\right)^2\left(\frac{6}{5}\right)$ ✓

$= \frac{\pi^2}{16} \cdot \frac{6}{5}$

$= \frac{3\pi^2}{40}$ ms^{-2} ✓

(b) What is the maximum speed? [2]

Max speed

$= kA$ ✓

$= \frac{\pi}{4} \times \frac{6}{5}$

$= \frac{3\pi}{10}$ ms^{-1} ✓

i.e. $v^2 = k^2(A^2 - x^2)$

Max speed occurs when $x = 0$

$\Rightarrow v^2 = k^2A^2$

$\Rightarrow v = kA$

$= \frac{\pi}{4} \times \frac{6}{5}$

$= \frac{3\pi}{10}$ ms^{-1}

2. (8 marks)

A particle moves in a straight line. Its displacement (metres) from a fixed point is given by $x(t)$ where t, is time in seconds. The acceleration of the particle is given by $a(t) = 2 - 4x$, where $x(0) = v(0) = 0$ and $v(t) \geq 0 \; \forall t$.

$\forall$ For all t.

(a) Determine v in terms of x. [4]

$$a(t) = v\frac{dv}{dx}$$

$$\Rightarrow 2 - 4x = v\frac{dv}{dx} \quad \checkmark$$

$$\Rightarrow \int (2-4x)\,dx = \int v\,dv$$

$$\Rightarrow 2x - 2x^2 = \frac{v^2}{2} + c \quad \checkmark$$

$$\Rightarrow v^2 = 4x - 4x^2 + k \qquad \text{where } k = 2c$$

$x(0) = v(0) = 0 \Rightarrow k = 0 \quad \checkmark$

$$\Rightarrow v = \sqrt{4x - 4x^2}$$

$$\therefore v = 2\sqrt{x - x^2} \quad \checkmark \quad \text{(discard } -2\sqrt{x-x^2}\text{)}$$

since $v(t) \geq 0, \forall t$.

(b) Hence, determine the range of values for x and v. [4]

$$x - x^2 \geq 0$$

$$\Rightarrow x(1-x) \geq 0$$

$$\therefore 0 \leq x \leq 1 \text{ m.}$$

$\checkmark \quad \checkmark$

Max value of $-x^2 + x$ occurs at $x = \frac{1}{2}$

$$\Rightarrow v = 2\sqrt{\tfrac{1}{2} - \tfrac{1}{4}}$$

$$= 1$$

$$\therefore 0 \leq v \leq 1 \text{ ms}^{-1}$$

$\checkmark \quad \checkmark$

3. (9 marks)

The time taken to complete a task has mean μ minutes and standard deviation 10 minutes. For Z as the standard normal variable, $P(-2.5 < Z < 2.5) \approx 0.988$.

(a) A sample of 100 students completed the task with a mean time of 102 minutes. State a 98.8% confidence interval for μ. [2]

$$102 \pm 2.5 \frac{10}{\sqrt{100}} \quad \checkmark$$

$$\Rightarrow \quad 99.5 \leq \mu \leq 104.5 \quad \checkmark \quad \text{minutes}$$

(b) Another sample of n students (where $n \geq 30$) is chosen. Determine n if we are to be 98.8% confident that the sample mean is to differ from μ by no more than 1.25 minutes. [3]

ie. an inequality

$$2.5 \frac{10}{\sqrt{n}} \leq 1.25 \quad \checkmark$$

$$\Rightarrow \quad \sqrt{n} \geq \frac{2.5 \times 10}{1.25} \quad \checkmark$$

$$= 20$$

$$\therefore \quad n = 400 \quad \checkmark$$

Population

(c) Given that $\mu = 100$ minutes, estimate the probability that a sample of 100 students will complete the task with a mean time exceeding 102.5 minutes. [4]

$$\bar{X} \sim N\left(100, \left(\frac{10}{\sqrt{100}}\right)^2\right) \sim N(100, 1^2) \quad \checkmark$$

$$P(\bar{X} > 102.5) = P\left(Z > \frac{102.5 - 100}{1}\right) \quad \checkmark$$

Recall: $z = \frac{x - \mu}{\sigma}$

$$= P(Z > 2.5)$$

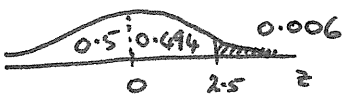
$$= P(Z > 0) - P(0 \leq Z \leq 2.5) \quad \checkmark$$

$$= 0.5 - \frac{0.988}{2}$$

$$= 0.5 - 0.494$$

$$= 0.006 \quad \checkmark$$

This diagram is dedicated to Trung!



End of Questions

TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 7 2016

Section 2 Calculator Assumed

Rectilinear Motion (including SHM) and Statistical Inference.

STUDENT'S NAME: SOLUTIONS

DATE: Thursday 8th September **TIME:** 30 minutes **MARKS**: 34

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet retained from Section 1.

Special Items: Drawing instruments, templates, three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment).

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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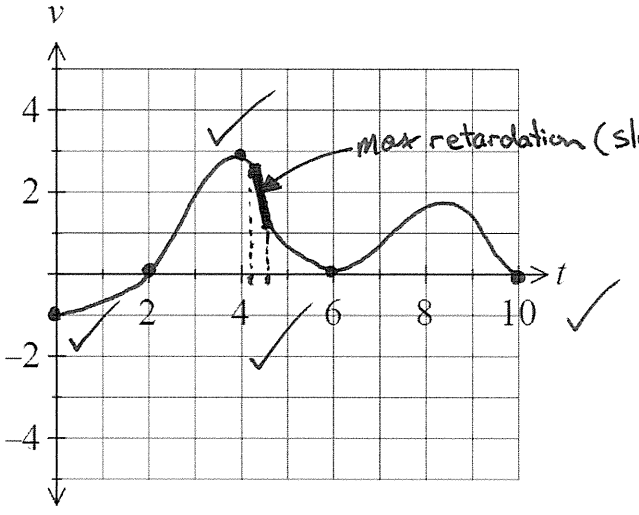
4. (7 marks)

An object, travelling in a straight line, has an initial velocity of $v = -1$ m/s and is capable of a maximum speed of 3 m/s. During its travels it reaches its maximum speed, stops twice but changes direction only once, before coming to rest after 10 seconds.

(a) Sketch on the axes below, a possible graph of $v = f(t)$ for $0 \le t \le 10$. [4]



Comment:
For all you 'straight' line people, I very nearly added in "with non-constant acceleration"

(b) Indicate on your graph when the object is subject to maximum retardation. [1]

ie. greatest negative gradient

(c) Provide a mathematical statement for calculating the distance travelled in the 10 seconds. [2]

Dist. Travelled = $\int_0^{10} |f(t)|\,dt$ ✓ ✓

5. (10 marks)

A vehicle travels along a straight stretch of highway. The driver notices a car stalled on the highway k meters ahead and applies the brakes of the vehicle. The acceleration of the vehicle t seconds after the breaks are applied is given by $a = -10e^{-0.1t}$

(a) Determine an expression for the displacement of the vehicle t seconds after the brakes are applied. [5]

$$v = \int -10e^{-0.1t}\,dt \quad \checkmark$$

$$= 100e^{-0.1t} + c \quad \checkmark$$

$$x = \int (100e^{-0.1t} + c)\,dt$$

$$= -1000e^{-0.1t} + ct + k \quad \checkmark$$

Given $x(0) = 0 \Rightarrow k = 1000$ ✓

$$\therefore x = -1000e^{-0.1t} + ct + 1000$$ ✓

(b) The vehicle comes to a complete stop after 3 seconds just behind the stalled car. Determine k and the initial speed of the vehicle. [5]

Given $v(3) = 0 \Rightarrow 100e^{-0.1(3)} + c = 0$ ✓

$\therefore c = -74.0818$ ✓

and $x = -1000e^{-0.1t} - 74.0818t + 1000$

now $k = x(3) = -1000e^{-0.1(3)} - 74.0818(3) + 1000$ ✓

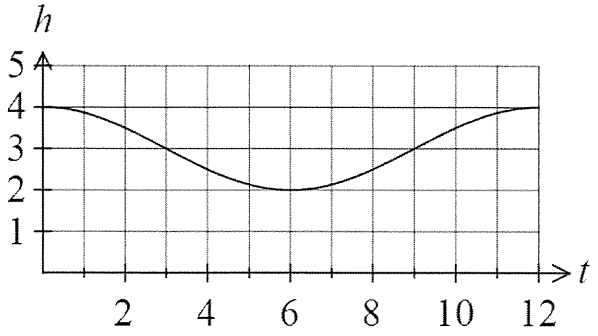
$= \underline{36.9\text{ m}}$ (1 d.p.) ✓

Initial speed of vehicle $v(0) = 100 - 74.0818$

$= \underline{25.9\text{ ms}^{-1}}$ (1 d.p.) ✓

6. (9 marks)

The depth of water $h(t)$ metres at a jetty is graphed against time (t hours) as shown in the accompanying diagram. The height, y, of the water surface above the mean water level satisfies the equation $\frac{d^2y}{dt^2} = -n^2 y$.



(a) State the depth of the mean water level. [1]

3m ✓

(b) Determine an expression for $y(t)$ and hence, determine an expression for $h(t)$ in terms of y(t). [3]

Let $y(t) = A\cos(kt+\beta)$ ✓

where $A=1$, $k=\frac{\pi}{6}$, $\beta = 0$

Recall:
Period: $T = \frac{2\pi}{k}$
$8 = \frac{2\pi}{k}$
$\therefore k = \frac{\pi}{6}$

$\therefore y(t) = \cos\left(\frac{\pi t}{6}\right)$ ✓

$\Rightarrow h(t) = 3 + \cos\left(\frac{\pi t}{6}\right)$ ✓

ie. $h(t) = 3 + y(t)$

(c) Calculate the time interval between two consecutive occasions when the water level is at a depth of 3.5 m. [2]

$3 + \cos\left(\frac{\pi t}{6}\right) = 3.5$ ✓

$\Rightarrow t = 2$ or $t = 10$ Use ClassPad

$\therefore$ the time interval is 8 hours apart ✓

(d) For 60% of the *period*, the water level exceeds k metres. Determine k. [3]

60% × 12

= 7.2 hrs (above)

$\Rightarrow$ 4.8 hrs below. ✓

$6 - 2.4 \leq t \leq 6 + 2.4$

$3.6 \leq t \leq 8.4$ ✓

$\therefore k = 3 + \cos\left(\frac{\pi \times 8.4}{6}\right)$

$= 2.69$ m (2 d.p.) ✓

7. (8 marks)

The time taken for a child to complete a particular puzzle is normally distributed* with a mean 3 minutes and standard deviation 20 seconds.

(a) A sample of fifty children of the same age collectively took 2 hours and 35 minutes to complete the puzzle. Calculate the mean time, in seconds, for this sample. [2]

$\bar{x} = \frac{155}{50}$ ✓

$= 3.1$ minutes

$= 186$ seconds ✓

(b) Estimate the probability that a second sample of 50 students of the same age will take a total of more than 2 hours and 35 minutes to complete the puzzle. [4]

By CLT $\bar{X} \sim N\left(180, \left(\frac{20}{\sqrt{50}}\right)^2\right)$ ✓✓

$P(\bar{x} \geq 186)$ ✓

$= 0.01695$ ✓

(c) Children who complete the puzzle under k seconds are classified 'highly gifted'. If 0.01% of all children are classified highly gifted, determine the value of k. [2]

* Population as given at the top of the page.

$X \sim N(180, 20^2)$ ✓

$P(\bar{x} < k) = 0.0001$

$\Rightarrow \quad k = 105.6$ sec. ✓

ie. Highly gifted complete puzzle in under 105.6 sec.

End of Questions